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The ‘exterior approach’: a new framework to solve inverse obstacle problems

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Abstract
We propose a new framework to solve inverse obstacle problems with a Dirichlet condition, consisting in the construction of a decreasing sequence of open domains that contain the searched obstacle. We provide a theoretical justification of this new methodology, infer from it a new algorithm based on the coupling of the quasi-reversibility technique and a level-set method, and illustrate the functionality of the algorithm with the help of numerical experiments in 2D.

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper, we address the inverse obstacle problem with a Dirichlet condition defined as follows. Let $\mathcal{D}$ be a bounded and connected open domain of $\mathbb{R}^d$ ($d \geqslant 2$) with the Lipschitz boundary, ν be the exterior normal to $\mathcal{D}$ and Γ be a non-empty open subset of $\partial\mathcal{D}$. For $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, $(g_0, g_1) \neq (0, 0)$, the inverse obstacle problem consists in finding a domain $\mathcal{O} \Subset \mathcal{D}$ with a continuous boundary such that $\Omega := \mathcal{D} \setminus \overline{\mathcal{O}}$ is connected and a function $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ that satisfy

$$\begin{cases} \Delta u &= 0 & \text{in } \Omega \\ u &= g_0 & \text{on } \Gamma \\ \partial_\nu u &= g_1 & \text{on } \Gamma \\ u &= 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{1}$$

Theorem 1.1. *The domain $\mathcal{O}$ and the function u that satisfy (1) are uniquely defined by the data* (g_0, g_1).

The proof of this theorem is well known and will therefore be omitted. It means that we can reasonably try to find the unknown obstacle $\mathcal{O}$ from the boundary Cauchy data (g_0, g_1) on Γ, which is the purpose of this paper. However, in practice the data are contaminated by some noise of amplitude δ, so we will have to cope with noisy data (g_0^δ, g_1^δ) instead of the exact data (g_0, g_1).

In this paper, we propose a new framework to solve the inverse obstacle problem with the Dirichlet condition. It relies on the following property.

Proposition 1. *Let* $(\omega_m)_{m\in\mathbb{N}}$ *be a sequence of open domains such that*

$$\forall m \in \mathbb{N},\ \mathcal{O} \subset \omega_{m+1} \subset \omega_m \Subset \mathcal{D}. \tag{2}$$

Then it converges, in the sense of the Hausdorff distance for open domains, to the set

$$\omega := \overbrace{\bigcap_{m\in\mathbb{N}} \omega_m}^{\circ},$$

such that $\mathcal{O} \subset \omega$.

Furthermore, if $u = 0$ *on* $\partial\omega$*, then* $\omega = \mathcal{O}$.

Proof. The fact that the sequence converges to an open set that contains the searched obstacle is a direct consequence of the stability of the convergence in the sense of the Hausdorff distance with respect to the inclusion and the decreasing nature of the sequence (see [34], section 2.2.3).

Now suppose that $u = 0$ on $\partial\omega$ and $\omega \neq \mathcal{O}$. Because $\mathcal{O} \subset \omega$, we have $\mathcal{R} := \omega \setminus \overline{\mathcal{O}} \neq \emptyset$. Furthermore, we have $\mathcal{R} \subset \Omega$, which implies that $u \in H^1(\mathcal{R}) \cap C^0(\overline{\mathcal{R}})$. Besides, we have $u = 0$ on $\partial\mathcal{O}$ and $u = 0$ on $\partial\omega$; hence $u = 0$ on $\partial\mathcal{R}$. From theorem 9.17 of [16], we obtain $u \in H^1_0(\mathcal{R})$. As by assumption we have $\Delta u = 0$ in $\mathcal{R}$, we obtain $u \equiv 0$ in $\mathcal{R}$, and by unique continuation $u = 0$ in Ω. This contradicts the fact that $(g_0, g_1) \neq 0$.

We conclude that $\omega \setminus \overline{\mathcal{O}} = \emptyset$, which leads to $\mathcal{O} \subset \omega \subset \overline{\mathcal{O}}$. Since $\mathcal{O}$ has a continuous boundary, we finally obtain $\omega = \mathcal{O}$. □

Proposition 1 invites us to build a method of identification of the obstacle based on the construction of a decreasing sequence of open domains, approaching the obstacle from the exterior. The exact solution u is used to define the sequence in order to ensure that $u = 0$ on the boundary of the limit open set. Therefore, the sequence will converge to the obstacle, and the inverse obstacle problem will be solved. We will refer to a method that relies on proposition 1 as an 'exterior approach' method.

In [13], we proposed an implementation of the 'exterior approach' based on the resolution of Poisson problems in the successive open domains. In this paper, we construct another 'exterior approach' based on the resolution of eikonal equations instead of Poisson problems. The eikonal equations widely appear in level-set methods, the fundamental idea of which is to represent a hypersurface, like an interface, through a function. This makes the computation of movements of this hypersurface significantly easier; in particular, the topology changes can be tracked straightforwardly. Since their introduction in [48], level-set techniques have been successfully used to solve inverse obstacle problems [52, 18, 3, 45]. In these articles the velocity of the eikonal equation is defined as the gradient of a suitably chosen cost function. These methods therefore rely on the optimization process. Our approach is different: the decreasing sequence of open sets is defined using the solutions of the eikonal equations with a velocity that depends directly on the function u. Thus, no optimization problem appears in its definition.

Many approaches have been proposed to solve obstacle problems. Some of them are based on shape sensitivity in the spirit of [53, 1], or topological gradients as in [21, 8], and so they rely on the optimization methods. In the particular case of the bi-dimensional Laplace equation, some methods based on conformal mappings exist, such as the one proposed in [32]. Another important class of algorithms are the (domain) sampling methods, such as the linear sampling method [24, 20], the factorization method [36, 37, 17], techniques based on the convex scattering support [40, 33] and the probe methods [35, 29] (see also [49] for a survey of these different methods). They try to determine if a given point (a set) is contained (contains) in the searched obstacle (a part of the searched obstacle). These methods typically either require more than a single pair of data or in general do not give an exact reconstruction of the obstacle; on the other hand, their assumptions on the searched obstacle are milder than that in this work. Recently, a new method named 'LASSO scheme' based on the probe methods was introduced in [50] to solve an inverse obstacle scattering problem. As the methods we developed in [13] and this paper, the 'LASSO scheme' builds a sequence of open sets $(\omega_m)_{m\in\mathbb{N}}$ that verifies

$$\mathcal{O} \subset \omega_{m+1} \subset \omega_m$$

and so it is a method that approaches the searched object from the exterior. Nevertheless, it is not an 'exterior approach' in the sense of this paper, as the construction of the sequence does not rely on the results of proposition 1. Furthermore, the 'LASSO scheme' uses a so-called front tracking method and an indicator function to construct ω_{m+1} from ω_m, whereas the 'exterior approach' uses level-set techniques and directly the function u to construct ω_{m+1} (see [46], chapter 3.1, for a comparison between the front tracking and level-set methods). To the best of our knowledge, the convergence of the sequence of open sets to the searched obstacle is only proved in [13] and this paper.

Among all these methods, the case of an obstacle characterized by a homogeneous Dirichlet condition has been studied e.g. in [32, 51, 39, 28] and [50].

Our paper is organized as follows. In section 2, we describe our new exterior approach method and provide its justification. Section 3 is devoted to the different techniques involved in its practical use. We particularly focus on the resolution of the ill-posed Cauchy problem by the quasi-reversibility method in the case of exact or noisy data. Numerical experiments are presented in section 4. We complete our study by a few concluding remarks in section 5.

2. An 'exterior approach' method based on eikonal equations

2.1. A heuristic presentation

Suppose that we have been able to construct m open sets $(\omega_k)_{0\leqslant k\leqslant m}$, such that

$$\mathcal{O} \subset \omega_m \subset \omega_{m-1} \subset \cdots \subset \omega_1 \subset \omega_0.$$

We want then to obtain an open set ω_{m+1} such that $\mathcal{O} \subset \omega_{m+1} \subset \omega_m$. To do so, we use a method inspired by level-set techniques: we define a function

$$\phi_m : (x,t) \in \mathcal{D} \times]0,T[\mapsto \phi_m(x,t) \in \mathbb{R},$$

with $T > 0$ being an arbitrary parameter, such that

$$\phi_m(x,0) < 0 \Leftrightarrow x \in \omega_m, \quad \phi_m(x,0) > 0 \Leftrightarrow x \in \mathcal{D} \setminus \overline{\omega}_m, \quad \phi_m(x,0) = 0 \Leftrightarrow x \in \partial\omega_m.$$

In other words, ϕ_m is a function of $x \in \mathcal{D}$ and an extra time-like parameter t such that ω_m is the open set where $\phi_m(x,0)$ is strictly negative. Then, the main idea of the method is to control the evolution of ϕ_m during the time interval $]0,T[$ such that, at final time T,

$$\mathcal{O} \subset \{x \in \mathcal{D},\ \phi_m(x,T) < 0\} \subset \omega_m. \tag{3}$$

Indeed, if (3) is verified, we can define ω_{m+1} as $\{x \in \mathcal{D},\ \phi_m(x,T) < 0\}$.

In the 'exterior approach' we build in this paper, ϕ_m is the solution of the eikonal equation

$$\frac{\partial \phi_m}{\partial t} - V_m|\nabla\phi_m| = 0 \quad \text{in } \mathcal{D} \times]0,T[\,, \tag{4}$$

so its evolution depends on the 'velocity' V_m. Therefore, we want to find sufficient conditions on V_m to guarantee (3) along with the convergence of the sequence to $\mathcal{O}$. These conditions are based on the following heuristic arguments: suppose first that we are able to build a 'smooth' velocity function V_m such that

$$\begin{cases} V_m = |u| \text{ outside } \omega_m \\ V_m \leqslant 0 \text{ inside } \mathcal{O}. \end{cases} \tag{5}$$

Let x be a point inside the obstacle. We have

$$\frac{\partial \phi_m}{\partial t}(x,t) = V_m(x)|\nabla\phi_m(x,t)| \leqslant 0$$

as $V_m(x) \leqslant 0$ by definition. Hence, $\phi_m(x,.)$ is a decreasing function, which implies that $\phi_m(x,T) \leqslant \phi_m(x,0) < 0$, that is,

$$\mathcal{O} \subset \{x \in \mathcal{D},\ \phi_m(x,T) < 0\}.$$

A similar reasoning shows that $\{x \in \mathcal{D},\ \phi_m(x,T) < 0\} \subset \omega_m$, and so relation (3) is verified.

Suppose that we build the sequence of decreasing open sets $(\omega_m)_{m\in\mathbb{N}}$ by iteratively solving equation (4) with a velocity V_m verifying (5) at each step. If we do so, the sequence $(\omega_m)_{m\in\mathbb{N}}$ will converge to a certain open set ω that contains $\mathcal{O}$. Let us give a heuristic proof that $\omega = \mathcal{O}$.

Suppose that there exists $x \in \partial\omega$ such that $u(x) \neq 0$. Then, as ω_m tends to ω, the velocities V_m of the eikonal equations will be positive in a neighborhood of x for sufficiently large m. This implies that $\frac{\partial\phi_m}{\partial t} = V_m|\nabla\phi_m|$ is positive in this neighborhood, which finally implies that $\phi_m(x,.)$ increases. If this increase remains significant during the iterations, we will have $\phi_m(x,T) \geqslant 0$ for a sufficiently large m, and then $x \notin \omega_m$, which contradicts the fact that $x \in \partial\omega$. Thus, $u(x) = 0$ for all $x \in \partial\omega$, and proposition 1 implies that $\omega = \mathcal{O}$.

Of course, this reasoning is not correct, but it gives a good view on how the method works. As it suggests, to prove the convergence of the method to the searched obstacle, we will have to control the temporal evolution of ϕ_m, that is, as $\frac{\partial\phi_m}{\partial t} = V_m|\nabla\phi_m|$, to control the value of the velocity functions and of the gradient of ϕ_m during the iterations.

In this section, we give a complete justification of the ideas we have developed in the introduction. The practical construction of velocities V_m that verifies (5) is addressed in section 3.

2.2. Some preliminary results

Let ω be an open domain such that $\omega \Subset \mathcal{D}$. We define $\eth_\omega$ the signed-distance function of ω by

$$\forall x \in \mathcal{D}, \quad \eth_\omega(x) = \operatorname{dist}(x,\overline{\omega}) - \operatorname{dist}(x,\mathcal{D}\setminus\overline{\omega}),$$

with dist being the standard Euclidean distance. The signed-distance function is 1-Lipschitz continuous and verifies $|\nabla\eth_\omega| = 1$ in the viscosity sense, and (see [26, 2])

$$\begin{cases} \eth_\omega(x) > 0 \Leftrightarrow x \in \mathcal{D}\setminus\overline{\omega} \\ \eth_\omega(x) < 0 \Leftrightarrow x \in \omega \\ \eth_\omega(x) = 0 \Leftrightarrow x \in \partial\omega. \end{cases}$$

Let V be an L-Lipschitz continuous function with compact support in $\mathcal{D}$. For $T > 0$, we define the following eikonal equation:

$$\begin{cases} \dfrac{\partial\phi}{\partial t} - V|\nabla\phi| = 0 & \text{in } \mathcal{D}\times\,]0,T[\\ \phi(x,0) = \eth_\omega(x) & \text{on } \mathcal{D} \\ \phi(x,t) = \eth_\omega(x) & \text{on } \partial\mathcal{D}\times\,]0,T[\,. \end{cases} \tag{6}$$

Proposition 2. *Equation (6) admits a unique viscosity solution ϕ, such that $\phi \in W^{1,\infty}(\mathcal{D} \times]0,T[)$. Therefore, ϕ verifies (6) almost everywhere.*

Proof. See [6]. □

The proof of our main theorem (theorem 2.1) relies on the following result.

Proposition 3. *Suppose that there exist $x \in \mathcal{D}$ and $\eta > 0$ such that $V \geqslant 0$ on $\mathcal{B}(x,\eta)$, the ball of the center x and radius η. Then ϕ, the solution of (6), verifies*

$$|\nabla\phi| \geqslant \exp\left(-\frac{5L}{2}t\right) \text{ a.e. in } \delta(x,\eta)$$

with L being the Lipschitz constant of V, C a positive constant s.t. $\sup_{\mathcal{D}} |V| \leqslant C$ *and*

$$\delta(x,\eta) := \left\{(y,t) \in \mathcal{B}(x,\eta) \times]0,T[\,,\, e^{Ct}(1+|y-x|) \leqslant 1+\eta\right\}.$$

Proof. A slight adaptation of [43], theorem 4.2 proves proposition 3 in the viscosity sense. Then the fact that $\phi \in W^{1,\infty}(\mathcal{D} \times]0,T[)$ and [6, corollary 2.1(ii)] show that it is true almost everywhere. □

2.3. Construction of the sequence

From now on, we suppose that u is a Lipschitz continuous function in $\overline{D}$.

Let ω_0 be an open domain such that $\mathcal{O} \subset \omega_0 \Subset \mathcal{D}$. Let $\chi \in C_c^\infty(\mathcal{D})$ such that $0 \leqslant \chi \leqslant 1$ in $\mathcal{D}$ and $\chi \equiv 1$ in ω_0. The function $\chi|u|$ is positive Lipschitz continuous in $\mathcal{D} \setminus \overline{\mathcal{O}}$ and $\chi(x)|u(x)| = 0$ for all $x \in \partial\mathcal{O}$.

Let V_0 be a Lipschitz continuous function in $\overline{D}$ such that

$$\begin{cases} V_0(x) = \chi(x)|u(x)| \text{ in } \mathcal{D} \setminus \overline{\omega}_0 \\ V_0(x) \leqslant 0 \text{ in } \mathcal{O}. \end{cases} \tag{7}$$

Such functions V_0 exist ($V_0 = \chi|u|$ in $\mathcal{D} \setminus \overline{\mathcal{O}}$ and $V_0 = 0$ in $\mathcal{O}$ satisfies (7)).

For a fixed $T > 0$, let ϕ_0 be the unique viscosity solution of the eikonal equation

$$\begin{cases} \dfrac{\partial\phi}{\partial t} - V_0|\nabla\phi| = 0 & \text{in } \mathcal{D} \times]0,T[\\ \phi(x,0) = \eth_{\omega_0}(x) & \text{in } \mathcal{D} \\ \phi(x,t) = \eth_{\omega_0}(x) & \text{on } \partial\mathcal{D} \times]0,T[\,. \end{cases} \tag{8}$$

As suggested in section 2.1, we define the open domain $\omega_1 := \{x \in \mathcal{D},\ \phi_0(x,T) < 0\}$.

Proposition 4. $\mathcal{O} \subset \omega_1 \subset \omega_0$.

Proof.

- First of all, we prove that $\mathcal{O} \subset \omega_1$. Let $x \in \mathcal{O}$ and $\eta > 0$ such that $\mathcal{B}(x,\eta) \subset \mathcal{O}$. As $\phi_0 \in W^{1,\infty}(\mathcal{D} \times]0,T[) \cap C^0(\overline{\mathcal{D}} \times [0,T])$, for all $y \in \mathcal{B}(x,\eta)$, $\phi_0(y,.) \in W^{1,\infty}(]0,T[)$, and we have (see [16, theorem 8.2])

$$\phi_0(y,T) - \phi_0(y,0) = \int_0^T \frac{\partial\phi_0}{\partial t}(y,t)\,\mathrm{d}t.$$

Hence, we have

$$\int_{\mathcal{B}(x,\eta)} \phi_0(y,T)\,\mathrm{d}y = \int_{\mathcal{B}(x,\eta)} \left(\phi_0(y,0) + \int_0^T \frac{\partial\phi_0}{\partial t}(y,t)\,\mathrm{d}t\right)\mathrm{d}y.$$

As ϕ_0 verifies equation (8) almost everywhere, we have

$$\int_{\mathcal{B}(x,\eta)} \phi_0(y,T)\,\mathrm{d}y = \int_{\mathcal{B}(x,\eta)} \left(\eth_{\omega_0}(y) + \int_0^T V_0(x)|\nabla\phi_0(y,t)|\,\mathrm{d}t \right) \mathrm{d}y.$$

Then, $\mathcal{B}(x,\eta) \subset \mathcal{O} \subset \omega_0$ implies $V_0 \leqslant 0$ in $\mathcal{B}(x,\eta)$, so

$$\int_{\mathcal{B}(x,\eta)} \phi_0(y,T)\,\mathrm{d}y \leqslant \int_{\mathcal{B}(x,\eta)} \eth_{\omega_0}(y)\,\mathrm{d}y.$$

Dividing by $|\mathcal{B}(x,\eta)|$ and letting η tend to zero, we obtain $\phi_0(x,T) \leqslant \eth_{\omega_0}(x) < 0$, that is, $x \in \omega_1$.

- Now, we prove that $\omega_1 \subset \omega_0$. Let $x \in \mathcal{D}$ such that $x \notin \omega_0$.
  - * *Either* $x \notin \overline{\omega}_0$. Let $\eta > 0$ such that $\mathcal{B}(x,\eta) \subset \mathcal{D} \setminus \overline{\omega}_0$. By definition, $\eth_{\omega_0} \geqslant 0$ and $V_0 \geqslant 0$ in $\mathcal{B}(x,\eta)$. As previously, we obtain
  $$\int_{\mathcal{B}(x,\eta)} \phi_0(y,T)\,\mathrm{d}y = \int_{\mathcal{B}(x,\eta)} \left(\eth_{\omega_0}(y) + \int_0^T V_0(x)|\nabla\phi_0(y,t)|\,\mathrm{d}t \right) \mathrm{d}y \geqslant 0.$$
  Dividing by $|\mathcal{B}(x,\eta)|$ and letting η tend to zero, we obtain $\phi_0(x,T) \geqslant 0$, which implies $x \notin \omega_1$.
  - * *Or* $x \in \partial\omega_0$. There exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \mathcal{D} \setminus \overline{\omega}_0$ and $x_n \xrightarrow{n\to\infty} x$. We just show that $\phi_0(x_n,T) \geqslant 0$. As ϕ_0 is a continuous function, we have $\phi_0(x_n,T) \xrightarrow{n\to\infty} \phi_0(x,T)$; hence $\phi(x,T) \geqslant 0$ and $x \notin \omega_1$. □

We now define the sequence $(\omega_m)_{m\in\mathbb{N}}$ by induction: for all $m \in \mathbb{N}$, we define

$$\omega_{m+1} := \{x \in \mathcal{D},\ \phi_m(x,T) < 0\},$$

with ϕ_m being the unique viscosity solution of the following eikonal equation:

$$\begin{cases} \dfrac{\partial\phi}{\partial t} - V_m|\nabla\phi| = 0 & \text{in } \mathcal{D} \times]0,T[\\ \phi_0(x,0) = \eth_{\omega_m}(x) & \text{on } \mathcal{D} \\ \phi(x,t) = \eth_{\omega_m}(x) & \text{on } \partial\mathcal{D} \times]0,T[\end{cases}$$

and V_m a Lipschitz continuous function that verifies

$$\begin{cases} V_m(x) = \chi(x)|u(x)|, \ \forall x \in \mathcal{D} \setminus \overline{\omega}_m \\ V_m \leqslant 0 \text{ on } \mathcal{O}. \end{cases}$$

Using the same arguments as in the proof of proposition 4, we prove that the sequence $(\omega_m)_{m\in\mathbb{N}}$ verifies $\mathcal{O} \subset \omega_{m+1} \subset \omega_m,\ \forall m \in \mathbb{N}$. Hence, according to proposition 1, it converges, in the sense of the Hausdorff distance, to an open domain ω such that $\mathcal{O} \subset \omega$. It remains to be proved that ω is the searched obstacle.

Remark. We can define V_m so that $V_m = \chi_m|u|$ outside ω_m, with $\chi_m \in C_c^\infty(\mathcal{D})$, $0 \leqslant \chi_m \leqslant 1$ and $\chi_m = 1$ in ω_m. But we are not obliged to do so: indeed, $\chi_m = \chi$ for all m is a relevant choice, as $\chi = 1$ on ω_0 and $\omega_m \subset \omega_0$ by construction.

2.4. Convergence of the sequence

Lastly, we state the main theorem of this section. In this view, we consider the following assumption:

[H] there exists $L > 0$ such that for all $m \in \mathbb{N}$, and for all $(x,y) \in \mathcal{D} \times \mathcal{D}$,

$$|V_m(x) - V_m(y)| \leqslant L\|x-y\|_{\mathbb{R}^d}.$$

Assumption [H] means that the functions V_m are uniformly Lipschitz continuous. The purpose of this assumption is to control the value of the velocities during the iterations in order to prove the convergence of the sequence to the obstacle as suggested in section 2.1.

Remark. One can easily verify that [H] implies the existence of $C > 0$ such that $\sup_{\mathcal{D}} |V_m| \leqslant C$ for all $m \in \mathbb{N}$.

Theorem 2.1. *Under assumption [H], the sequence $(\omega_m)_{m\in\mathbb{N}}$ converges, in the sense of the Hausdorff distance for open domains, to the obstacle $\mathcal{O}$.*

According to proposition 1, we only need to prove that $u = 0$ on $\partial\omega$. We need the following proposition (see [34, proposition 2.2.14]):

Proposition 5. *Let $x \in \partial\omega$. There exists $(x_m)_{m\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ such that $\forall m \in \mathbb{N}$, $x_m \in \partial\omega_m$ and $x_m \xrightarrow{m\to\infty} x$.*

First, we prove that if there exists $x \in \partial\omega$ such that $u(x) \neq 0$, then there exists a neighborhood of x where, for sufficiently large m, the velocity functions V_m are strictly positive.

Lemma 2.2. *Suppose [H] and there exists $x \in \partial\omega$ such that $u(x) \neq 0$. Then there exist $M > 0$, $\eta > 0$ and $\alpha > 0$ such that $\forall m \geqslant M$, $\forall y \in \mathcal{B}(x, \eta)$, $V_m(y) \geqslant \alpha$.*

Proof. Since $u(x) \neq 0$, there exists $\alpha > 0$ such that $|u(x)| > 3\,\alpha$. As u is a continuous function, there exists $\eta_1 > 0$ such that $|u| \geqslant 3\,\alpha$ in $\mathcal{B}(x, \eta_1)$.

Proposition 5 implies that, for all $m \in \mathbb{N}$, there exists $x_m \in \partial\omega_m$ such that x_m tends to x. By definition, $V_m(x_m) = |u(x_m)|$ for all m. Furthermore, for m being sufficiently large we have $x_m \in \mathcal{B}(x, \eta_1)$, which implies

$$V_m(x_m) = |u(x_m)| \geqslant 3\alpha. \tag{9}$$

Assumption [H] implies that $\forall (y, z) \in \mathcal{D} \times \mathcal{D}$, $|V_m(y) - V_m(z)| \leqslant L|y - z|$. For $\eta := \frac{\alpha}{L}$, hence, we have

$$V_m(y) \geqslant V_m(z) - L|y - z| \geqslant V_m(z) - \alpha, \ \forall y \in \mathcal{D}, \ \forall z \in \mathcal{B}(y, \eta) \cap \mathcal{D}. \tag{10}$$

Now, let $\eta_2 := \min(\eta, \eta_1)$. For m being sufficiently large, we have $x_m \in \mathcal{B}(x, \eta_2) \subset \mathcal{B}(x, \eta_1)$, hence (9) leads to $V_m(x_m) \geqslant 3\alpha$. Furthermore, $x_m \in \mathcal{B}(x, \eta_2)$ implies $x \in \mathcal{B}(x_m, \eta_2) \subset \mathcal{B}(x_m, \eta)$; hence (10) leads to $V_m(x) \geqslant V_m(x_m) - \alpha \geqslant 2\,\alpha$. Finally, (10) implies that for m sufficiently large, for all $y \in \mathcal{B}(x, \eta)$, we have

$$V_m(y) \geqslant V_m(x) - \alpha \geqslant \alpha.$$

□

Now, we prove that if there exists $x \in \partial\omega$ such that $u(x) \neq 0$, we can control the decrease of the gradient of ϕ_m in a neighborhood of x for sufficiently large m.

Lemma 2.3. *Suppose [H] and there exists x on $\partial\omega$ such that $u(x) \neq 0$. Then there exist $M > 0$ and $\eta > 0$ such that*

$$|\nabla\phi_m| \geqslant \exp(-\gamma t), \ \textit{a.e. in } \mathcal{B}\left(x, \frac{\eta}{2}\right) \times]0, T_0[\,, \ \forall m \geqslant M,$$

with

$$\gamma := \frac{5L}{2}, \quad T_0 := \frac{1}{C} \ln\left(\frac{2 + 2\eta}{2 + \eta}\right),$$

where L is the Lipschitz constant of V_m and C such that $\sup_{\Omega} |V_m| \leqslant C$.

Proof. Lemma 2.2 shows that there exist $M > 0$ and $\eta > 0$ such that $V_m \geqslant 0$ in $\mathcal{B}(x, \eta)$ for $m \geqslant M$. By using proposition 3, we obtain

$$|\nabla\phi_m| \geqslant \exp(-\gamma t)$$

almost everywhere in $\delta(x, \eta) := \left\{(y, t) \in \mathcal{B}(x, \eta) \times]0, T[\,, \ e^{Ct}(1 + |y - x|) \leqslant 1 + \eta\right\}$.

Let $(y, t) \in \mathcal{B}(x, \frac{\eta}{2}) \times]0, T_0[$. We have

$$e^{Ct}(1 + |y - x|) \leqslant e^{CT_0}\left(1 + \frac{\eta}{2}\right) \leqslant \frac{2 + 2\eta}{2 + \eta}\left(1 + \frac{\eta}{2}\right) \leqslant 1 + \eta.$$

The result follows. □

We now prove theorem 2.1, showing that the existence of $x \in \partial\omega$ such that $u(x) \neq 0$ leads to a contradiction.

Proof of theorem 2.1. Suppose there exists $x \in \partial\omega$ such that $u(x) \neq 0$. There exists $x_n \in \omega$ such that x_n tends to x. By definition of ω, for all $m \in \mathbb{N}$, $\phi_m(x_n, T) < 0$, which implies $\phi_m(x, T) \leqslant 0$ as $\phi_m(., T)$ is a continuous function.

Using lemmas 2.2 and 2.3, we obtain that for all ε , $0 < \varepsilon < \frac{\eta}{2}$, for all $m \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathcal{B}(x,\varepsilon)} [\phi_m(y, T) - \phi_m(y, 0)]\, \mathrm{d}y &= \int_{\mathcal{B}(x,\varepsilon)} \int_0^T \frac{\partial\phi_m}{\partial t}(y, s)\, \mathrm{d}s\, \mathrm{d}y \\ &= \int_{\mathcal{B}(x,\varepsilon)} \int_0^T V_m(y)|\nabla\phi_m(y, s)|\, \mathrm{d}s\, \mathrm{d}y \\ &\geqslant \alpha \int_{\mathcal{B}(x,\varepsilon)} \int_0^T |\nabla\phi_m(y, s)|\, \mathrm{d}s\, \mathrm{d}y \\ &\geqslant \alpha \int_{\mathcal{B}(x,\varepsilon)} \int_0^{T_0} |\nabla\phi_m(y, s)|\, \mathrm{d}s\, \mathrm{d}y \\ &\geqslant \alpha \int_{\mathcal{B}(x,\varepsilon)} \int_0^{T_0} \exp(-\gamma s)\, \mathrm{d}s\, \mathrm{d}y.\end{aligned}$$

Hence, we obtain

$$\int_{\mathcal{B}(x,\varepsilon)} [\phi_m(y, T) - \phi_m(y, 0)]\, \mathrm{d}y \geqslant \frac{\alpha}{\gamma}\left[1 - \exp(-\gamma T_0)\right]|\mathcal{B}(x, \varepsilon)|.$$

Dividing by $|\mathcal{B}(x, \varepsilon)|$, and letting ε go to zero, we obtain

$$\frac{\alpha}{\gamma}[1 - \exp(-\gamma T_0)] \leqslant \phi_m(x, T) - \phi_m(x, 0) = \phi_m(x, T) - \eth_{\omega_m}(x) \leqslant -\eth_{\omega_m}(x). \tag{11}$$

We know that there exists a sequence $x_m \in \partial\omega_m$ such that $x_m \xrightarrow{m\to\infty} x$ (proposition 5). So, by definition of the signed-distance function $\eth_{\omega_m}$, we have

$$|\eth_{\omega_m}(x)| = |\eth_{\omega_m}(x) - \eth_{\omega_m}(x_m)| \leqslant |x - x_m| \xrightarrow{m\to\infty} 0.$$

Therefore, the right-hand side of (11) tends to zero as m goes to infinity, and we obtain $\frac{\alpha}{\gamma}\left[1 - \exp(-\gamma T_0)\right] \leqslant 0$, which is impossible. Thus, $u = 0$ on $\partial\omega$, and the theorem is proven. □

We have built a theoretical 'exterior approach' based on the resolution of eikonal equations: we have obtained a suitable sequence of open sets $(\omega_m)_{m\in\mathbb{N}}$ that converges from the exterior to the searched obstacle. We can infer from this theoretical approach a practical algorithm to solve the inverse obstacle problem: starting from an initial open domain ω_0 sufficiently large such that it contains the searched obstacle, we will, at each step of our process

(i) compute u outside ω_m from the possibly noisy boundary condition (g_0, g_1). In other words, we want to solve the Cauchy problem: find u that verifies

$$\begin{cases} \Delta u &= 0 & \text{in } \mathcal{D} \setminus \overline{\omega}_m \\ u &= g_0 & \text{on } \Gamma \\ \partial_\nu u &= g_1 & \text{on } \Gamma. \end{cases}$$

The Cauchy problem is well known to be severely ill posed. Consequently, we need a regularization method to solve it, and obtain a quasi-solution u_ε close to the exact solution u.

(ii) Compute the speed function V_m of the eikonal equation by extending $|u|$ inside ω_m, so that V_m is a Lipschitz continuous function with a Lipschitz constant independent of m, and verifies $V_m < 0$ in the unknown obstacle.

(iii) Compute $\eth_{\omega_m}$, the signed-distance function of ω_m, and solve the eikonal equation.

The following section is devoted to the study of these three different aspects of the method.

Remark. It should be noted that despite the use of u_ε in place of u in the definition of the sequence $(\omega_m)_{mM\in\mathbb{N}}$, the sequence remains a decreasing one; hence it still converges to an open set ω. Nevertheless, theorem 2.1 no longer holds, and we cannot ensure that ω is the unknown obstacle. To estimate the discrepancy between ω and $\mathcal{O}$ is a challenging and difficult question.

Remark. In [15], we apply the 'exterior approach' framework to the non-destructive testing of elastic–plastic media from boundary measurements. The idea is to find defects (cracks), thanks to the plastic zones they create. In the antiplane case, the bi-dimensional problem consists in finding, in an elastic–plastic medium $\mathcal{D}$, an open domain $\mathcal{P}$ (the plastic zone) and a function u such that

$$\begin{cases} \Delta u &= 0 & \text{in } \mathcal{D} \setminus \overline{\mathcal{P}} \\ u &= g_0 & \text{on } \Gamma \\ \partial_\nu u &= g_1 & \text{on } \Gamma \\ |\nabla u| &< 1 & \text{in } \mathcal{D} \setminus \overline{\mathcal{P}} \\ |\nabla u| &= 1 & \text{on } \partial \mathcal{P}, \end{cases}$$

with (g_0, g_1) measurements on Γ, open subset of $\partial\mathcal{D}$. We show that a slight modification of the 'exterior approach' based on the resolution of Poisson problems, briefly described in [13], allows us to solve this inverse problem.

To use the 'exterior approach' based on eikonal equations to solve this problem, one has to simply modify the definition of functions V_m by

$$V_m = 1 - |\nabla u|^2 \text{ in } \mathcal{D} \setminus \overline{\omega}_m, \quad V_m < 0 \text{ in } \mathcal{O},$$

and then build the sequence $(\omega_m)_{m\in\mathbb{N}}$ as above. By construction, the sequence converges to an open domain ω that contains the searched plastic zone, and one can easily verify that under assumption [H], $1 - |\nabla u|^2 = 0$ on $\partial\omega$, that is, $\omega = \mathcal{P}$.

3. Main aspects of the process

3.1. Regularization of the ill-posed Cauchy problem using the quasi-reversibility method

In this section, $\omega \Subset \mathcal{D}$ is an open domain such that $\Omega := \mathcal{D} \setminus \overline{\omega}$ is connected. The domain ω plays the part of any element of the sequence $(\omega_m)_{m\in\mathbb{N}}$. Assume that $u \in H^2(\Omega)$ solves the following problem:

$$\begin{cases} \Delta u = 0 & \text{in} & \Omega \\ u = g_0 & \text{on} & \Gamma \\ \partial_\nu u = g_1 & \text{on} & \Gamma. \end{cases}$$

This problem is well known to be severely ill posed, as it admits at most one solution that does not depend continuously on the data, and may have no solution. From now on, we suppose that it admits a (unique) solution. Several regularization methods to solve this problem exist (see for example [7, 5, 4, 22]). Most of them rely on an optimization process, which is a drawback for our purpose: indeed, we have to solve a Cauchy problem at each step of our method, and it would be extremely computationally costly to solve an optimization problem at each iteration. For this reason, we use a more 'direct' method to solve this problem: the so-called quasi-reversibility method, first introduced in [42], and then revisited in [9, 10, 38]. We now describe this method first in the case of exact data, and then in the case of noisy data.

We first introduce the two following sets:

$$V := \left\{v \in H^2(\Omega),\ v_{|\Gamma} = g_0,\ \partial_\nu v_{|\Gamma} = g_1\right\}$$
$$V_0 := \left\{v \in H^2(\Omega),\ v_{|\Gamma} = 0, \partial_\nu v_{|\Gamma} = 0\right\}.$$

V is a non-empty set as $u \in V$. It is also clear that V_0, endowed with the classic H^2 scalar product, is a Hilbert space. We now introduce the following variational formulation of quasi-reversibility.

Problem(QR) For $\varepsilon > 0$, find $u_\varepsilon \in V$ such that for all $v \in V_0$, we have

$$(\Delta u_\varepsilon, \Delta v)_{L^2(\Omega)} + \varepsilon(u_\varepsilon, v)_{H^2(\Omega)} = 0. \tag{12}$$

Using the Lax–Milgram theorem, one can easily prove that problem (QR) admits a unique solution $u_\varepsilon \in V$.

Theorem 3.1. *The solution of the quasi-reversibility problem u_ε tends to the solution of the Cauchy problem u as ε tends to 0, and we have the estimate*

$$\|\Delta u_\varepsilon - \Delta u\|_{L^2(\Omega)} \leqslant \sqrt{\varepsilon}\|u\|_{H^2(\Omega)}.$$

Proof. Taking $v = u_\varepsilon - u \in V_0$ in (12) and using that $\Delta u = 0$ in Ω, we obtain

$$(\Delta u_\varepsilon, \Delta u)_{L^2(\Omega)} + \varepsilon(u_\varepsilon, u)_{H^2(\Omega)} = 0. \tag{13}$$

Hence, we have $(u_\varepsilon, u_\varepsilon - u)_{H^2(\Omega)} \leqslant 0$, which leads to $\|u_\varepsilon\|_{H^2(\Omega)} \leqslant \|u\|_{H^2(\Omega)}$. Subtracting $\varepsilon(u, u_\varepsilon - u)_{H^2(\Omega)}$ from equation (13), we obtain

$$\|\Delta u_\varepsilon - \Delta u\|^2_{L^2(\Omega)} + \|u_\varepsilon - u\|^2_{H^2(\Omega)} = -\varepsilon(u, u_\varepsilon - u)_{H^2(\Omega)},$$

and therefore

$$\|u_\varepsilon - u\|^2_{H^2(\Omega)} \leqslant |(u, u_\varepsilon - u)_{H^2(\Omega)}| \leqslant \|u\|_{H^2(\Omega)}\|u_\varepsilon - u\|_{H^2(\Omega)}$$

and we finally obtain $\|u_\varepsilon - u\|_{H^2(\Omega)} \leqslant \|u\|_{H^2(\Omega)}$. Now, equation (13) leads to the estimate

$$\|\Delta u_\varepsilon - \Delta u\|^2_{L^2(\Omega)} \leqslant \varepsilon|(u_\varepsilon, u_\varepsilon - u)_{H^2(\Omega)}| \leqslant \varepsilon\|u\|^2_{H^2(\Omega)}. \tag{14}$$

Let $(\varepsilon_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ such that $\varepsilon_n \xrightarrow{n\to\infty} 0$. We define $u_n := u_{\varepsilon_n}$. As seen previously, we have $\|u_n\|_{H^2(\Omega)} \leqslant \|u\|_{H^2(\Omega)}$; hence, we can extract a subsequence $u_{n'}$ that weakly converges to $w \in H^2(\Omega)$. Since Δ is a linear continuous operator on $H^2(\Omega)$, it is weakly continuous, and we have $\Delta w = \lim_{n'\to\infty} \Delta u_{n'} = \Delta u$. For the same reason, we also have

$$w_{|\Gamma} = \lim_{n'\to\infty} u_{n'|\Gamma} = g_0, \qquad \partial_\nu w_{|\Gamma} = \lim_{n'\to\infty} \partial_\nu u_{n'|\Gamma} = g_1.$$

The uniqueness of the solution of the Cauchy problem implies that $w = u$, that is, $u_{n'} \rightharpoonup u$. Finally, we have

$$\|u_{n'} - u\|^2_{H^2(\Omega)} \leqslant |(u, u_{n'} - u)_{H^2(\Omega)}| \xrightarrow{n'\to\infty} 0,$$

and the sequence $u_{n'}$ converges to u strongly in $H^2(\Omega)$. One can then easily prove that $u_\varepsilon \xrightarrow{\varepsilon\to 0} u$ strongly in $H^2(\Omega)$. □

To obtain a convergence rate for the quasi-reversibility method is a delicate issue. For problems posed in $\mathbb{R}^2$ or $\mathbb{R}^3$, a logarithmic convergence rate can be obtained using stability results for the Cauchy problem [11, 12].

One can solve problem (QR) in $\mathbb{R}^2$ using finite difference schemes as in [42], or splines as in [23]. But these two methods of discretization require simple geometries of computation, which cannot be guaranteed in our method. Therefore, we prefer to discretize the problem using a finite element method based on the so-called Fraeijs de Veubeke elements (F.V.1 elements). These nonconforming finite elements were initially introduced in [41] to solve plate bending problems, and their utilization in the quasi-reversibility method has been studied in [13], including a convergence analysis. The discrete formulation of the quasi-reversibility problem is

Problem(QRh) For $\varepsilon > 0$, find $u_{\varepsilon,h} \in V_{h,g}$ such that for all $v_h \in V_{h,0}$,

$$\sum_{T\in\mathcal{T}_h} (\Delta u_{\varepsilon,h}, \Delta v_h)_{L^2(T)} + \varepsilon(u_{\varepsilon,h}, v_h)_{H^2(T)} = 0.$$

Here, $V_{h,g}$ (resp. $V_{h,0}$) is the discrete F.V.1 approximation of V (resp. V_0). Problem (QRh) is well posed, and we have the following convergence estimate, assuming that u_ε is smooth enough:

$$\|u_{\varepsilon,h} - u_\varepsilon\|_h := \sqrt{\sum_{T\in\mathcal{T}_h} \|u_{\varepsilon,h} - u_\varepsilon\|^2_{H^2(T)}} \leqslant C\frac{h}{\varepsilon},$$

where the constant C is independent of h. Therefore, we have an efficient finite element method to solve the Cauchy problems that appear in the exterior approach in the case of exact data.

Now suppose that we have some noisy data $(g_0^\delta, g_1^\delta) \in L^2(\Gamma) \times L^2(\Gamma)$, such that $\|g_i^\delta - g_i\|_{L^2(\Gamma)} \leqslant \delta_i$. Because of the severe instability of the Cauchy problem and the non-smoothness of the noisy data, we cannot use them as a limit condition for the quasi-reversibility method. Furthermore, it is well known that we must choose the parameter of regularization ε functions of the amplitude of noise δ. In [14], we propose a method based on the Morozov principle and the duality in optimization to fulfil the double objective of regularizing the noisy data and setting the parameter of regularization. First, for $\alpha \geqslant 0$, we define the functional set

$$V_\alpha^\delta := \{v \in H^2(\Omega),\ \|\Delta v\|_{L^2(\Omega)} \leqslant \alpha,\ \|v_{|\Gamma} - g_0^\delta\|_{L^2(\Gamma)} \leqslant \delta,\ \|\partial_\nu v_{|\Gamma} - g_1^\delta\|_{L^2(\Gamma)} \leqslant \delta\}$$

which is a non-empty set as $u \in V_\alpha^\delta$. We introduce the following optimization problem.

Problem(P_α) Find $u_\alpha^\delta \in V_\alpha^\delta$ such that $\frac{1}{2}\|u_\alpha^\delta\|^2_{H^2(\Omega)} = \inf_{v\in V_\alpha^\delta} \frac{1}{2}\|v\|^2_{H^2(\Omega)}$.

For all $\alpha \geqslant 0$, problem (P_α^δ) has a unique solution. Function $u^\delta := u_0^\delta$ is the harmonic function of minimal H^2 norm such that $u^\delta_{|\Gamma}$ and $\partial_\nu u^\delta_{|\Gamma}$ approach the noisy data g_0^δ and g_1^δ up to the noise amplitude δ: it is a Morozov-type approximation of the solution of the Cauchy problem. As we cannot hope to reconstruct u from the noisy data, it is from now on our reference solution. We have the following proposition.

Proposition 6.

$$\lim_{\delta\to 0} \|u^\delta - u\|_{H^2(\Omega)} = 0, \quad \lim_{\alpha\to 0} \|u_\alpha^\delta - u^\delta\|_{H^2(\Omega)} = 0.$$

We introduce the problem (P_α^*), dual optimization problem of (P_α) (for more information on duality in optimization, see for example [27]).

Problem(P_α^*) Find $p_\alpha^\delta = (p_{\alpha 0}^\delta, p_{\alpha 1}^\delta, p_{\alpha 2}^\delta) \in Y := L^2(\Gamma) \times L^2(\Gamma) \times L^2(\Omega)$ such that

$$G_\alpha^\delta(p_\alpha^\delta) = \inf_{p \in Y} G_\alpha^\delta(p),$$

where, for all $p = (p_0, p_1, p_2) \in Y$,

$$G_\alpha^\delta(p) = \frac{1}{2}\|A^* p\|^2_{H^2(\Omega)} + \delta_0\|p_0\|_{L^2(\Gamma)} - (g_0^\delta, p_0)_{L^2(\Gamma)} + \delta_1\|p_1\|_{L^2(\Gamma)} - (g_1^\delta, p_1)_{L^2(\Gamma)} + \alpha\|p_2\|_{L^2(\Omega)}$$

and

$$A := v \in H^2(\Omega) \mapsto (v_{|\Gamma}, \partial_\nu v_{|\Gamma}, \Delta v) \in Y.$$

Problem (P_α^*) is an unconstrained optimization problem, so it is easier to solve it than problem (P_α).

Proposition 7. *For $\alpha > 0$, problem (P_α^*) admits a unique solution p_α^δ, and*

$$u_\alpha^\delta = A^* p_\alpha^\delta.$$

Propositions 6 and 7 show that in order to obtain a good approximation of u^δ, we can solve problem (P_α^*) with a small parameter α. Remark that the choice of parameter α is independent of the noise amplitude δ, allowing us to choose it as small as we want. In [14], numerical experiments show that $\alpha = 10^{-4}$ is a relevant choice.

Finally, we state our main theorem.

Theorem 3.2. *For $\alpha > 0$, define $(\tilde{g}_0^\delta, \tilde{g}_1^\delta) := (u_{\alpha|\Gamma}^\delta, \partial_\nu u_{\alpha|\Gamma}^\delta) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\varepsilon_\alpha^\delta := \dfrac{\alpha}{\|p_{\alpha 2}^\delta\|_{L^2(\Omega)}}$. Then u_α^δ is the unique solution of the following quasi-reversibility problem:*

Find $u \in H^2(\Omega)$, $(u_{|\Gamma}, \partial_\nu u_{|\Gamma}) = (\tilde{g}_0^\delta, \tilde{g}_1^\delta)$, such that for all $v \in H^2(\Omega)$, $(v_{|\Gamma}, \partial_\nu v_{|\Gamma}) = (0, 0)$,

$$(\Delta u, \Delta v) + \varepsilon_\alpha^\delta (u, v)_{H^2(\Omega)} = 0.$$

This theorem means that we have fulfilled our double objective: indeed, we have obtained $(\tilde{g}_0^\delta, \tilde{g}_1^\delta)$ smooth enough to be used in the quasi-reversibility method and a value $\varepsilon_\alpha^\delta$ of the parameter of regularization dependent on the level of noise δ. Furthermore, we show in [14] that $\varepsilon_\alpha^\delta$ is almost the best choice of regularization parameter, in the sense that it almost minimizes the discrepancy between u, the solution of the Cauchy problem, and u_ε^δ, the solution of the quasi-reversibility problem with regularized data $(\tilde{g}_0^\delta, \tilde{g}_1^\delta)$. Finally, the study of the discretization of problem (P_α^*) on the space of finite elements F.V.1 and its resolution using the limited memory BFGS algorithm [44, 19] was carried out in [14], including various numerical examples. We invite interested readers to consult this article for additional details.

Finally, in the 'exterior approach method', we proceed as follows. In the presence of noisy data, we solve at the first step of the algorithm the optimization problem (P_α^*) in the (small) domain $\mathcal{D} \setminus \overline{\omega_0}$ in order to regularize the data and to obtain a suitable parameter of regularization. Then, we solve the Cauchy problem in $\mathcal{D} \setminus \overline{\omega_m}$ using the quasi-reversibility method with these regularized data and the parameter of regularization. In the presence of exact data, we use directly the quasi-reversibility method with these data and an arbitrary small parameter ε.

3.2. Extension of $|u|$

To extend a velocity only defined on a part of a domain to the whole domain of interest is a standard problem in the level-set framework. Indeed, level-set methods are often used to track the motion of an interface: in some cases a relevant velocity is only defined on this interface, whereas these methods need the definition of the velocity everywhere in the domain of study. A typical example is the simulation of the motion of a fire front: the velocity is physically defined only on the fire front. For this reason, many methods have been developed in the level-set community in order to extend a velocity, most of them relying on the resolution of a Hamilton–Jacobi equation (see [46, chapter 8]).

In our 'exterior approach' method, we need to construct the velocity V_m, extension of $|u|$ inside the current open domain ω_m, such that

(a) $V_m < 0$ in the obstacle and
(b) V_m is a Lipschitz continuous function, with the Lipschitz constant being independent of m.

Let us show that (a) can be fulfilled by solving a Poisson equation inside ω_m.

First, we define $U \in L^2(\mathcal{D})$ such that

$$\begin{cases} U = |u| & \text{in } \mathcal{D} \setminus \overline{\mathcal{O}} \\ U \in H_0^1(\mathcal{O}) & \\ U \leqslant 0 & \text{in } \mathcal{O}. \end{cases} \tag{15}$$

Such a function exists (take $U = |u|$ outside $\mathcal{O}$ and $U = 0$ inside $\mathcal{O}$) and is an element of $H^1(\mathcal{D})$. Consequently, ΔU is an element of $H^{-1}(\mathcal{D})$. Let $f \in H^{-1}(\mathcal{D})$ be such that

$$f > \Delta U \ \text{(in the sense of distributions)}. \tag{16}$$

We consider the following problem defined in the current open set ω_m.

Problem(P) Find $v_m \in H^1(\omega_m)$ such that $\begin{cases} \Delta v_m & = & f & \text{in } \omega_m \\ v_m & = & |u| & \text{on } \partial\omega_m. \end{cases}$

Problem (P) has a unique solution, and we have:

Proposition 8. *The solution of problem (P) is negative in the searched obstacle.*

Proof. As $\mathcal{O} \subset \omega_m$, we have by definition $U = |u|$ on $\partial\omega_m$, and the limit condition $v_m = |u|$ on $\partial\omega_m$ is equivalent to $v_m - U \in H_0^1(\omega_m)$. Furthermore, we have $\Delta v_m - \Delta U = f - \Delta U > 0$ in ω_m. The weak maximum principle (see [30]) implies that $v_m < U$ in ω_m. As by definition $U \leqslant 0$ in $\mathcal{O}$, and by construction $\mathcal{O} \subset \omega_m$, the result follows. □

If we define

$$V_m := \begin{cases} |u| \text{ outside } \omega_m \\ v_m \text{ inside } \omega_m, \end{cases}$$

with v_m being the solution of problem (P) for f 'sufficiently large', that is f verifying (16), V_m verifies directly (a). Furthermore, if we choose f as a constant, v_m is smooth by elliptic regularity property, so V_m is a Lipschitz continuous function. However, it seems difficult to construct V_m in a way that ensures its Lipschitz constant does not depend on m.

From a practical point of view, the main issue is the condition that f must be 'sufficiently large'. As we do not know u and $\mathcal{O}$, we are not able to construct any function U that verifies (15), and so we cannot ensure *a priori* that f verifies (16) (even if we know that such f exists). Therefore, f appears as a parameter of the method that we will have to choose properly. The practical choice of parameter f and its influence on the reconstruction of the obstacle are addressed section 4.1.2.

Remark. The limit condition $v_m = |u|$ on $\partial\omega_m$ makes sense only if ω_m has a Lipschitz boundary. As in the numerical two-dimensional experiments we solve the Poisson problem in a domain union of some triangles of a mesh, it will always be the case.

To solve problem (P) in a two-dimensional polygonal domain Θ_m, we use a standard Lagrange finite element approach: let $\mathcal{T}_h$ be a regular triangulation of Θ_m, and denote by V_h^1 the space generated by P_1 Lagrange finite elements, and by $V_{h,0}^1$ the subspace of functions in V_h^1 that vanish on $\partial\Theta_m$. We introduce the discrete variational problem.

Problem(Ph) Find $v_{m,h} \in V_h^1$, such that $v_{m,h}(x) = |u(x)|$ for all $x \in \partial\Theta_m$ vertices of $\mathcal{T}_h$, and

$$\int_{\Theta_m} \nabla v_{m,h}.\nabla w_h \,\mathrm{d}x = -\int_{\Theta_m} f w_h \,\mathrm{d}x, \qquad \forall w_h \in V_{h,0}^1.$$

Problem (Ph) has a unique solution $v_{m,h}$ that converges to v_m as h goes to zero. As in the applications we do not know u but its discrete quasi-reversibility approximation $u_{\varepsilon,h}$, we replace the limit condition of problem (Ph) by $v_{m,h}(x) = |u_{\varepsilon,h}(x)|$ for all $x \in \partial\Theta_m$ vertices of $\mathcal{T}_h$.

3.3. Resolution of the eikonal equation

The final step of the algorithm consists in solving the eikonal equation

$$\begin{cases} \dfrac{\partial\phi}{\partial t} - V_m|\nabla\phi| &= 0 & \text{in } \mathcal{D}\times]0,T[\\ \phi(x,t) &= \eth_{\omega_m}(x) & \text{on } \partial\mathcal{D}\times]0,T[\\ \phi(x,0) &= \eth_{\omega_m}(x) & \text{on } \mathcal{D}, \end{cases}$$

with $T > 0$ being an arbitrary constant and $\eth_{\omega_m}$ the signed-distance function of ω_m. Such an equation appears in lots of mathematical domains, and its numerical resolution is well known. We do not go into detail about the method we use to solve the equation, and we refer to [46, 47] for a important overview on such methods.

In this paper, we solve the eikonal equation using a standard finite difference ENO-scheme. We obtain the signed-distance function $\eth_{\omega_m}$ by solving the following Hamilton–Jacobi problem:

$$\frac{\partial\phi}{\partial t} + S_{\omega_m}(|\nabla\phi| - 1) = 0, \quad S_{\omega_m}(x) = \begin{cases} 1 & \text{if } x \notin \overline{\omega}_m \\ -1 & \text{if } x \in \omega_m \\ 0 & \text{if } x \in \partial\omega_m \end{cases}$$

using a finite difference method (see [46, chapter 7], which is dedicated to the construction of the signed-distance function).

4. Numerical experiments

In this section, we present the results of numerical experiments in dimension 2. The domain $\mathcal{D}$ is the square $]-0.5, 0.5[\times]-0, 5, 0, 5[$ and the non-convex obstacle $\mathcal{O}$ is the union of the two discs of respective centers $(-0.2, 0)$ and $(0.23, 0.2)$ and radii 0.15 and 0.1 (see figure 1).

Given an open subset Γ of $\partial\mathcal{D}$, our experiments are based on artificial Cauchy data (g_0, g_1) obtained by solving the following Laplace problem for a given $\tilde{g}_1 \in H^{-1/2}(\partial\mathcal{D})$:

$$\begin{cases} \Delta u &= 0 & \text{in } \mathcal{D}\setminus\overline{\mathcal{O}} \\ \partial_\nu u &= \tilde{g}_1 & \text{on } \partial\mathcal{D} \\ u &= 0 & \text{on } \partial\mathcal{O} \end{cases}$$

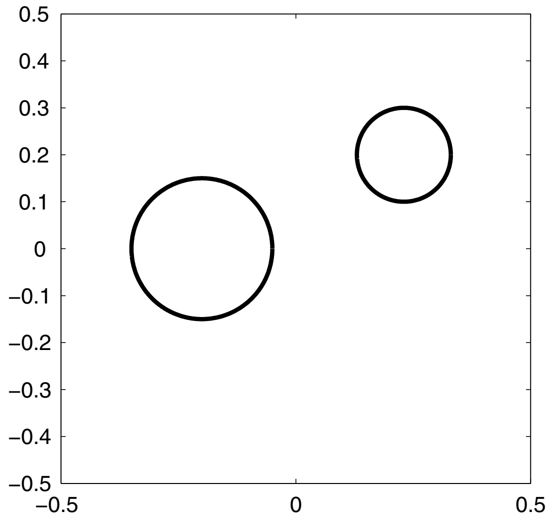

Figure 1. The obstacle.

and defining $(g_0, g_1) := (u_{|\Gamma}, \tilde{g}_{1|\Gamma})$. In this paper, (g_0, g_1) are based on $\tilde{g}_1$ defined by

$$\begin{cases} \tilde{g}_1 = 1 \text{ on }]-0.5, 0.5[\times \{-0.5\} \cup]-0.5, 0.5[\times \{0.5\} \\ \tilde{g}_1 = 0 \text{ on } \{-0.5\} \times]-0.5, 0.5[\cup \{0.5\} \times]-0.5, 0.5[\,. \end{cases}$$

Since the boundaries of $\mathcal{D}$ and $\mathcal{O}$ are respectively Lipschitz and smooth, we have $u \in H^2(\mathcal{D}\backslash\overline{\mathcal{O}})$ (see [31]) as required in the previous sections.

In our algorithm, we solve problems (QRh) and (Ph) using finite element methods, and the eikonal equation using a finite difference scheme. To solve all these problems on a single mesh, we build the mesh by first dividing $\mathcal{D}$ in squares of size h, and then dividing these squares in two triangles to obtain a structured regular triangulation $\mathcal{T}_h$. In the following experiments, $h = 10^{-2}$. The complete algorithm we use to solve the inverse obstacle problem is the following.

Algorithm

- Initialization
  - (–) Choose an initial guess Θ_0 as the union of triangles of $\mathcal{T}_h$ such that $\mathcal{O} \subset \Theta_0$ and $\mathcal{D} \setminus \overline{\Theta_0}$ is connected.
  - (–) In the case of noisy data, solve optimization problem (P^*_α) in $\mathcal{D} \setminus \overline{\Theta_0}$ to regularize the data and set the parameter of regularization ε to be used in the quasi-reversibility method. Otherwise, set $\varepsilon := 10^{-5}$.
- Iterations
  - (i) The polygonal domain Θ_m being given, solve the discrete quasi-reversibility problem (QRh) in $\mathcal{D} \setminus \overline{\Theta_m}$. The solution is denoted $u_{\varepsilon,h,m}$.
  - (ii) Solve the discrete Poisson problem (Ph) using $|u_{\varepsilon,h,m}|$ as the Dirichlet limit condition. The solution is denoted $v_{h,m}$.
  - (iii) Define the velocity function $V_{h,m}$ of the eikonal equation as $V_{h,m} := |u_{\varepsilon,h,m}|$ in $\mathcal{D} \setminus \overline{\Theta_m}$, $V_{h,m} := v_{h,m}$ in Θ_m.

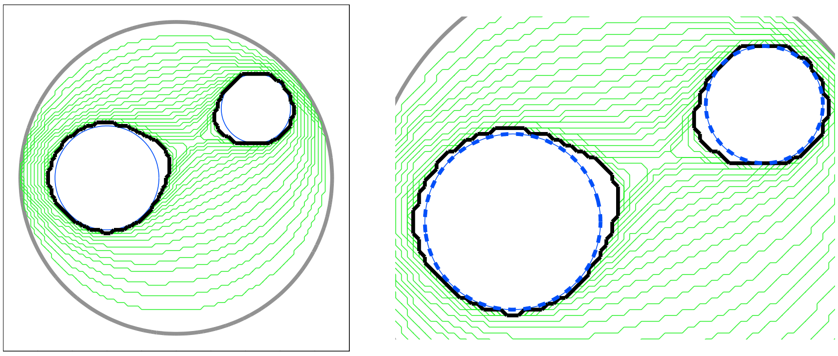

Figure 2. Identification of the obstacle.

(iv) Compute the signed-distance function of Θ_m and solve the eikonal equation in $\mathcal{D}\times]0,1[$, the solution of which is denoted $\phi_{h,m}$. Define

$$\omega_{m+1} := \left\{x \in \mathcal{D},\ \phi_{h,m}(x,1) < 0\right\}$$

and Θ_{m+1} as the union of the triangles of $\mathcal{T}_h$ having at least one vertex in ω_{m+1}.

(v) Go back to step (i) until the stopping criterion is reached.

Some remarks about this algorithm.

- The whole method uses a single mesh. Thus, we can precompute the different matrices at the beginning of the algorithm leading to a significative saving of time during the process.
- In initialization 2: the value $\varepsilon = 10^{-5}$ in the case of exact data comes from the study of the quasi-reversibility method in [25], where we show that $\varepsilon = 10^{-5}$ provides a good quality of reconstruction and a low condition number for the quasi-reversibility finite element matrix.
- In iterations: we define Θ_m as an 'exterior approximation' of ω_m consisting of the union of triangles of the mesh. Indeed, ω_m is not required to be the union of triangles of the mesh, which would be a liability since we want to solve finite element problems both outside and inside ω_m. An easy way to get rid of this problem is to replace ω_m with Θ_m. Of course, it introduces a step back of maximum size h in the process, which can be considered as a drawback, but it has also an interesting advantage: doing this, one can easily show that there exists $m \in \mathbb{N}$ such that for all $p \in \mathbb{N}$, $\Theta_{m+p} = \Theta_m$. In other words, the open sets Θ_m stop moving forward after some iterations. This means that we can stop the algorithm as soon as the Hausdorff distance between two successive domains is zero. This gives us a strong stopping criterion (see section 4.1.1).

Remark that Θ_m can be equivalently defined as the union of triangles of the mesh that have at least one vertex A such that $\phi_{m-1}(A,1) < 0$. Therefore, the construction of Θ_m is immediate once we know ϕ_{m-1}.

4.1. Case of complete exact data

In this section, we present results obtained for $\Gamma := \partial\mathcal{D}$ and exact data (g_0, g_1). The result of identification of the obstacle is presented in figure 2 for f, second member of Poisson problem (Ph), equal to 8, and for initial guess Θ_0 delimited by a polygonal curve that approximates the circle of center (0,0) and radius 0,45.

Figure 3 shows the results of the algorithm with different initial guesses Θ_0, all things being equal. Clearly, the initial guess does not influence the reconstruction of the obstacle.

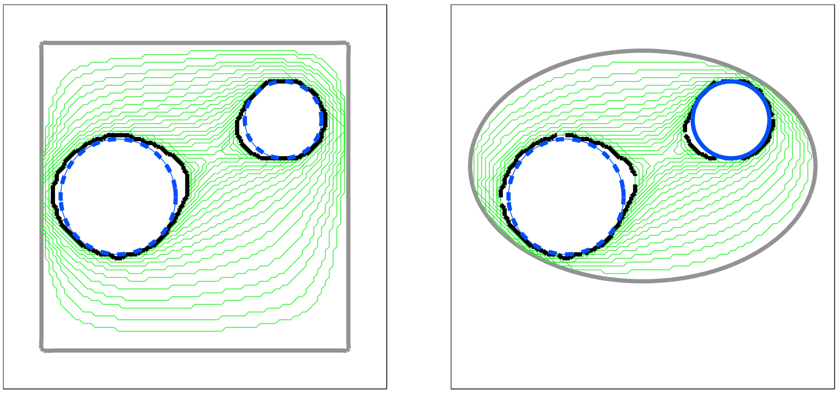

Figure 3. Reconstruction of the obstacle for different initial guesses: square and ellipse.

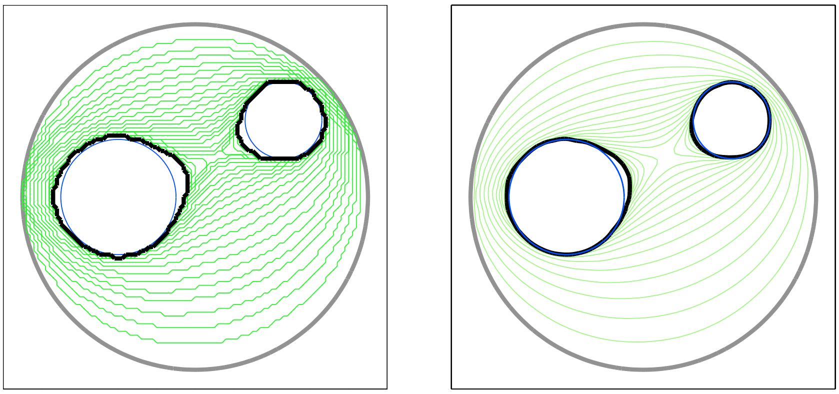

Figure 4. Comparison between the two 'exterior approach' methods.

Therefore, we always choose Θ_0 as the disc of center (0,0) and radius 0,45 in the following numerical experiments.

Finally, in figure 4 the reconstruction obtained with the 'exterior approach' based on the resolution of the eikonal equations described in this paper and the one obtained with the 'exterior approach' based on the resolution of Poisson problems described in [13] are compared. The results are very close.

4.1.1. Stopping criterion. Figure 5 depicts the Hausdorff distance between Θ_m and the obstacle and between Θ_m and Θ_{m+1} functions of the number of iterations m. The change of slope around the 20th iteration in the left figure corresponds to the division of Θ_m in two connected components. The right figure clearly shows that stopping the algorithm whenever the Hausdorff distance between two successive open domains is exactly zero is a robust stopping criterion, as mentioned previously.

4.1.2. Influence and choice of parameter f. In figure 6, we study the influence of f, second member of the Poisson problem, on the quality of the reconstruction of the unknown obstacle. As expected, a smaller f (here $f = 5$) leads to a better reconstruction of the obstacle.

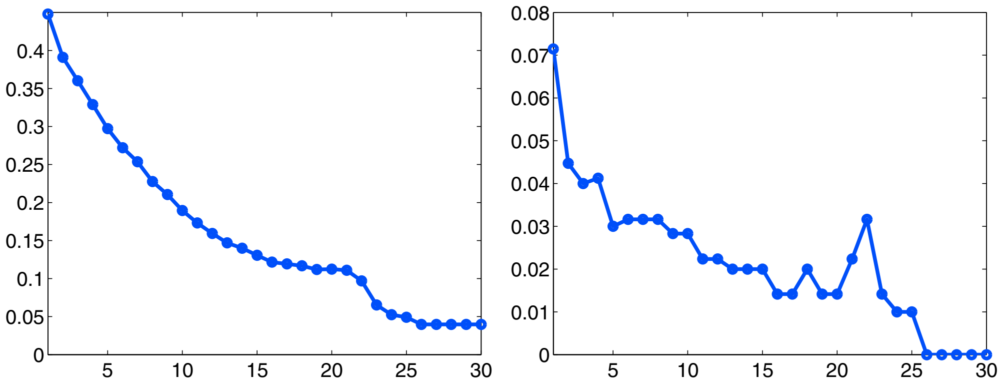

Figure 5. $d_H(\Theta_m, \mathcal{O})$ and $d_H(\Theta_m, \Theta_{m+1})$ functions of m.

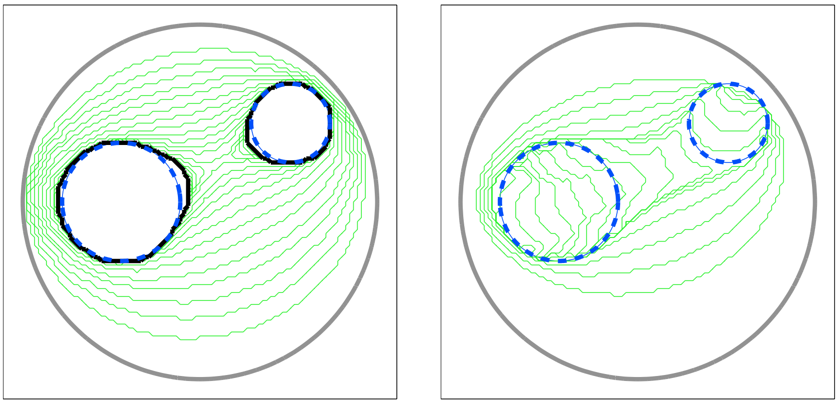

Figure 6. Influence of the parameter f: left $f = 5$, right $f = 2$.

Nevertheless, we see that the algorithm does not converge if we choose f too small (here $f = 2$), which is consistent with the condition that f must be chosen 'sufficiently large' (condition (16)). Therefore, we can set the parameter f by first choosing f relatively large, then decreasing the parameter until the algorithm no longer converges.

Nevertheless, the search for the optimal f makes sense only if the exact data are available. Indeed, the noise on data affects the reconstruction of the obstacle in such a way that it is unclear whether parameter f has some values that are better than others. For this reason, in the following experiments, we simply choose $f = 8$, without trying any optimization of the parameter.

4.2. Case of complete noisy data

In this section, we present results of identification for $\Gamma := \partial\mathcal{D}$ and noisy data (g_0^δ, g_1^δ). Noisy data are obtained by adding to the exact data a Gaussian noise, given by a standard normal distribution, such that

$$\|g_i^\delta - g_i\|_{L^2(\Omega)} \leqslant \sigma \|g_i\|_{L^2(\Omega)}, \quad \forall i \in \{0, 1\}.$$

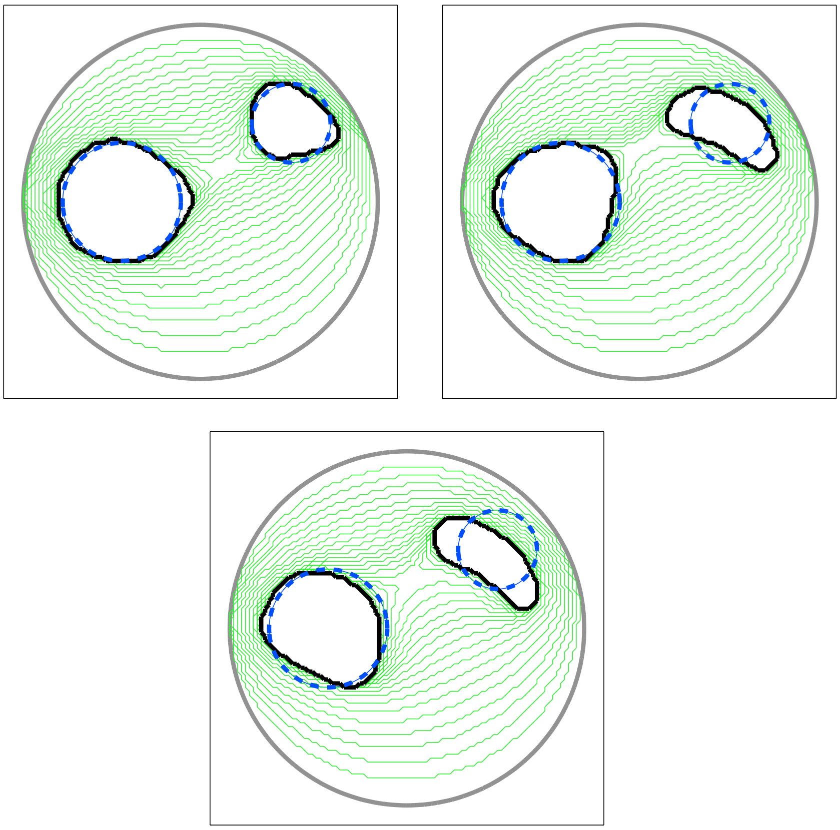

Figure 7. Identification of the obstacle with noisy data: $\sigma = 1\%,\ 5\%,\ 10\%$.

Parameter σ is the relative amplitude of noise. As explained in section 3, in such a case we solve the optimization problem (P^*_α) in $\mathcal{D} \setminus \overline{\Theta}_0$ in order to obtain regularized data and the parameter of regularization we use in the quasi-reversibility method the following steps. Figure 7 depicts some results of identification for different values of the noise amplitude σ.

4.3. Case of partial noisy data

The use of the quasi-reversibility method in our algorithm allows us to consider partial Cauchy data, that is, Γ is strictly included in $\partial\mathcal{D}$. In figure 8, we present the results of identification for partial noisy data, with a 2% relative noise. The support of the Cauchy data is specified in bold in the figures.

5. Concluding remarks

The 'exterior approach' we have presented in this paper provides a new and simple framework to solve inverse obstacle problems. This framework leads to efficient practical algorithms: a first one that couples the quasi-reversibility method and the resolution of Poisson problems is presented in [13], and another that couples the quasi-reversibility method and the resolution of eikonal equations is proposed in this paper.

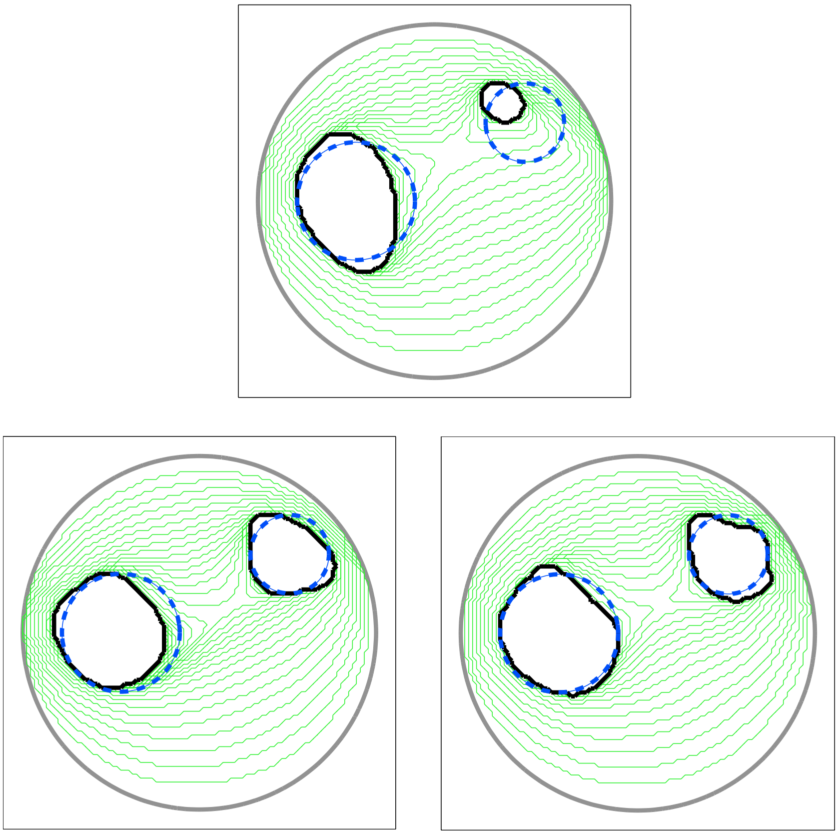

Figure 8. Identification of the obstacle with partial noisy data: $\sigma = 2\%$.

Thanks to the use of the quasi-reversibility, these algorithms do not rely on the resolution of an optimization problem in the case of exact data, and rely on the resolution of an optimization problem posed on a small part of the domain in the case of noisy data, which differentiates our approach from the classical ones. In the presence of noisy data, the use of the duality-based optimization method we have proposed in [14] to regularize the data and set the parameter of regularization gives good results in our opinion, particularly when the amplitude of noise is severe. Furthermore, thanks to the quasi-reversibility, the algorithms handle easily the situations where there exists a subpart of the boundary on which no data are available ($\Gamma \subsetneq \partial\mathcal{D}$).

On the other hand, the level-set approach we use to build the sequence of open sets allows topology changes during the process, without adding technical difficulties. The coupling of quasi-reversibility and level-set methods is achieved using a single mesh, which leads to a significative saving of the computation time.

Furthermore, the 'exterior approach' is easily modifiable to solve inverse problems with obstacles characterized by a limit condition that depends only on the function u, for example $|\nabla u| = c$. However, applying this method to the inverse obstacle problem with a Neumann condition ($\partial_\nu u = 0$ on the boundary of the searched obstacle) is not straightforward, and would require further developments.

Finally, we would like to emphasize that the use of the quasi-reversibility method to obtain u outside the current open domain ω_m and the resolution of the Poisson problem to extend

$|u|$ inside ω_m are technical choices we have made to implement the 'exterior approach' based on the resolution of eikonal equations. Indeed, any method to regularize the Cauchy problem could be used in the first step of the algorithm, and any method that suitably constructs the velocity V_m could be used in the second step.

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